Name:____________________

Class:____________________

James Ruse Agricultural High School

2024 **YEAR 12 Trial HSC Examination**

Mathematics Extension 2

General Instructions:

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black pen.
- Only calculators approved by NESA may be used.
- A reference sheet is provided.
- In Questions 11–15, show relevant mathematical reasoning and calculations.

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10.
- Allow approximately 15 minutes for this section.

Section II – 90 marks (pages 6–10)

- Attempt Questions 11–15.
- Allow approximately 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1-10.
Allow approximately 15 minutes for this section.
Write your answers on the multiple-choice answer sheet provided.

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1. A complex number z is defined such that $|z - 1 + 2i| = 1$. Which of the following is the maximum modulus of z?

   (A) $2\sqrt{2}$

   (B) $\sqrt{5} + 1$

   (C) $\sqrt{5}$

   (D) $2\sqrt{2} + 1$

2. Which of the following is the magnitude of the vector $\cos\theta\, \underset{\sim}{i} + \sin\theta\, \underset{\sim}{j} + \tan\theta\, \underset{\sim}{k}$, where $0 < \theta < \frac{\pi}{2}$?

   (A) 1

   (B) $\operatorname{cosec}\theta$

   (C) $\cot\theta$

   (D) $\sec\theta$

3. Suppose John found a raven, and it was black. Which of the following **must** be false?

   (A) "There exist non-black ravens as well as black ravens."

   (B) "There exist non-black ravens."

   (C) "All ravens are black."

   (D) None of the above.

4. The points A, B and C are collinear where

$$\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}, \qquad \overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}, \qquad \overrightarrow{OC} = 3\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}$$

   Which of the following are the values of a and b?

   (A) $a = -3,\ b = -2$

   (B) $a = 3,\ b = -2$

   (C) $a = -3,\ b = 2$

   (D) $a = 3,\ b = 2$

5. Let $z = \sqrt{3} + i$. Which of the following gives the geometric effect of multiplying the complex number w by $\frac{\bar{z}}{z}$?

(A) w is rotated anticlockwise by an angle of $\frac{\pi}{3}$.

(B) w is rotated anticlockwise by an angle of $\frac{2\pi}{3}$.

(C) w is rotated clockwise by an angle of $\frac{\pi}{3}$.

(D) w is rotated clockwise by an angle of $\frac{2\pi}{3}$.

6. Let A and B be true statements such that $A \Rightarrow B$. Which of the following statements is necessarily false?

(A) $\sim B \Rightarrow \sim A$

(B) A and $\sim B$

(C) A or $\sim B$

(D) None of the above.

7. It is given that $z = 1 + i$ is a root of $z^3 + bz^2 + 6z - 4 = 0$, where b is a real number. Which of the following is the value of b?

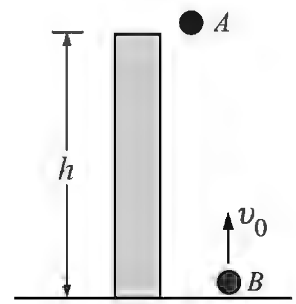
(A) -4

(B) 4

(C) 2

(D) -2

8. The diagram shows ball A being dropped from rest at time $t = 0$ seconds from a tower of height h metres. At the same instant, ball B is launched upward from the ground with initial speed v_0. If air resistance is negligible, and assuming all measurements are with respect to the centre of mass of each object, which of the following gives the best approximation for the time at which the balls pass each other?



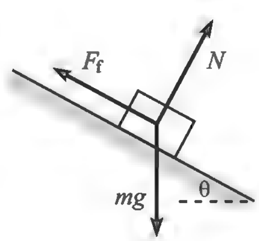
(A) $\frac{2h}{v_0}$

(B) $\frac{h}{v_0}$

(C) $\frac{h}{2v_0}$

(D) $\frac{h}{4v_0}$

9. The diagram below shows a block at rest on a plane that is inclined at an angle of measure θ. The forces acting on the mass are gravitational, normal and frictional forces, as indicated. The forces are NOT depicted to scale. Which of the following statements is always true?



(A) $N \leq mg,\ F_f \leq mg$

(B) $N \geq mg,\ F_f \leq mg$

(C) $N \leq mg,\ F_f \geq mg$

(D) None of the above.

10. Consider the integral

$$\int_{-3}^{x^2-3x} e^{t^2}\,dt$$

where x is a variable limit. Which of the following values of x minimises the integral?

(A) $\frac{1}{2}$

(B) $\frac{3}{2}$

(C) $\frac{5}{2}$

(D) None of the above.

Section II begins on the next page.

Section II

90 marks
Attempt Questions 11-15.
Allow approximately 2 hours and 45 minutes for this section.
Start each question on a new page. Extra paper is available.
In Questions 11-15, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (18 marks) **Start a new page**.

(a) Consider the complex numbers $a = 3 - 5i$ and $b = 2 + 3i$. Evaluate $a - \bar{b}$. **1**

(b) Solve for $z \in \mathbb{C}$, **3**

$$z^2 + 2\bar{z} + 3 = 0$$

(c) Use the substitution $u = \tan x$ to evaluate **3**

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x \, dx$$

(d) By using an appropriate substitution, find **2**

$$\int x^3 \sqrt{1 + x^2} \, dx$$

(e) Let $n \in \mathbb{Z}$ with $n \geq 0$. Define

$$I_n = \int_0^{\pi} x^n \sin x \, dx$$

(i) Show that **3**

$$I_n = \pi^n - n(n-1)I_{n-2}$$

(ii) Hence evaluate **2**

$$\int_0^{\pi} x^6 \sin x \, dx$$

leaving your answer in exact form.

(f) A particle has velocity equation

$$\dot{x}^2 = 4x - x^2$$

(i) Show that the particle is undergoing simple harmonic motion. **1**
(ii) Find the centre of motion, amplitude and period of the motion. **2**
(iii) If the displacement is $x = 4$ at time $t = 0$, find the displacement x as a function of time t. **1**

Question 12 (18 marks) **Start a new page**.

(a) On an Argand diagram, sketch the region representing the set of all z satisfying $2 < |z| \leq 4$ and $\frac{\pi}{6} \leq \arg(z) < \frac{\pi}{3}$, showing the coordinates of any vertices. **3**

(b) If $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are unit vectors such that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$, find the value of $\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{a}$. **2**

(c) A sphere $\mathcal{S}_1$, with centre $C(2, 2, 2)$ passes through the point $A(4, 4, 4)$.

(i) Find the Cartesian equation of $\mathcal{S}_1$. **2**

(ii) A second sphere $\mathcal{S}_2$ has equation $(x-2)^2 + (y-2)^2 + (z-5)^2 = 1$. Find the equation of the circle in which $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect. **2**

(d) (i) Show that $a^2 + b^2 \geq 2ab$. **1**

(ii) Hence or otherwise, show that for all positive numbers a, b and c, **2**

$$2(a^3 + b^3 + c^3) \geq ab(a+b) + bc(b+c) + ca(c+a)$$

(e) Suppose a particle is moving horizontally, measured by a coordinate system. Initially, the particle is at the origin O and moving with velocity 2 ms^{-1}. The acceleration of the particle is given by $\ddot{x} = x - 2$ where x is its displacement in metres at time t, measured in seconds.

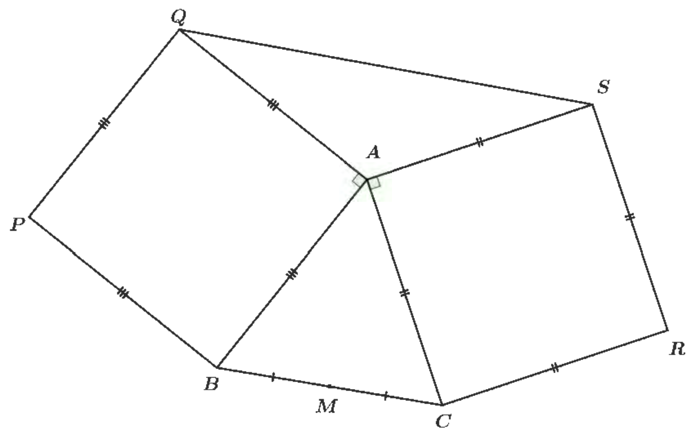
(i) Show that the velocity of the particle is given by $\dot{x}^2 = (x-2)^2$. **2**

(ii) Explain why $0 \leq x < 2$ for all $t \in [0, \infty)$ and hence show that $\dot{x} = 2 - x$. **2**

(iii) Find an expression for x as a function of t. **2**

Question 13 (18 marks) **Start a new page.**

(a) Let ABC be a triangle and let M be the midpoint of BC. Squares $ABPQ$ and $ACRS$ are erected on sides AB and AC as shown in the diagram below. Prove, using vector methods, that $|\overrightarrow{QS}| = 2|\overrightarrow{AM}|$. **3**



(b) Consider the two perpendicular vectors $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(i) Find the unit vectors $\hat{\underset{\sim}{u}}$ and $\hat{\underset{\sim}{v}}$. **2**

(ii) Find a unit vector $\hat{\underset{\sim}{w}}$ that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

(iii) Let the vector $\underset{\sim}{x} = a\hat{\underset{\sim}{u}} + b\hat{\underset{\sim}{v}} + c\hat{\underset{\sim}{w}}$ where a, b, c are scalar constants. If $|\underset{\sim}{x}| = 1$, prove that $a^2 + b^2 + c^2 = 1$. **2**

(iv) Let α, β and γ be the measure of the angles between $\hat{\underset{\sim}{u}}$ and $\underset{\sim}{x}$, $\hat{\underset{\sim}{v}}$ and $\underset{\sim}{x}$, $\hat{\underset{\sim}{w}}$ and $\underset{\sim}{x}$ respectively. Prove that $\cos\alpha + \cos\beta + \cos\gamma = a + b + c$. **2**

(c) (i) Given ω is a non-real root of $x^3 - 1 = 0$, show that ω is also a root of $x^2 + x + 1 = 0$. **1**

(ii) Hence prove by contradiction that $(x+1)^{2n} + x^{2n} + 1$ is not divisible by $(x^2 + x + 1)$ if n is divisible by 3, where x is non-real. **2**

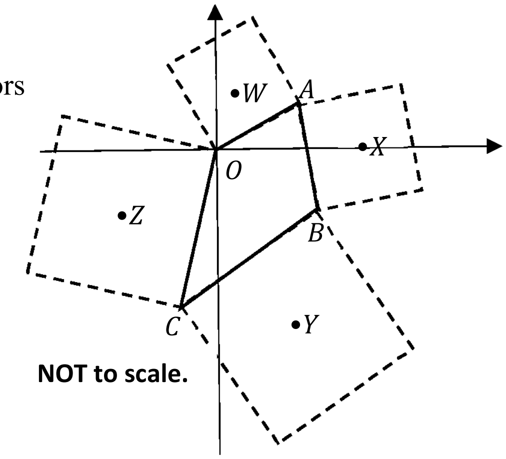
(d) Prove by mathematical induction that **4**

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta = \frac{\sin\left(\frac{2n-1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

for all $n \geq 2, n \in \mathbb{Z}$ and $\theta \neq 2k\pi$ for all $k \in \mathbb{Z}$.

Question 14 (18 marks) **Start a new page.**

(a) Let $OABC$ be a convex quadrilateral where the vectors $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CO}$ correspond to $2a$, $2b$, $2c$ and $2d$ respectively, where $a, b, c, d \in \mathbb{C}$.



The centres of the squares erected on each side of $OABC$ are W, X, Y and Z as shown in the diagram.

(i) Show that X is represented by the complex number $2a + b + ib$. **2**

(ii) Show that the line segments WY and XZ are equal in length and perpendicular. **3**

(b) Consider the following number sequence given by: **4**

$$a_1 = 0$$

$$a_n = \frac{1 + a_{n-1}}{2 + a_{n-1}} \text{ for } n \in \mathbb{Z}, n \geq 2$$

Prove, by mathematical induction, that $a_{n-1} < a_n$ for all integers $n \geq 2$.

(c) (i) Prove, by the method of partial fractions, that **2**

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

(ii) Hence, or otherwise, use integration by parts to find **2**

$$\int \frac{\tan^{-1} x}{x^2} dx$$

(d) (i) Show that, for $x \in \mathbb{R}$ such that $|x| < 1$, **1**

$$\sum_{k=0}^{n} (-1)^k x^k = \frac{1}{1+x} - \frac{(-1)^{n+1} x^{n+1}}{1+x}$$

(ii) Prove that **2**

$$0 \leq \int_0^1 \frac{x^{n+1}}{1+x} dx \leq \frac{1}{n+2}$$

(iii) By integrating part (i) and using the result in part (ii), show that **2**

$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k}{k+1} = \ln 2$$

Question 15 (18 marks) **Start a new page**.

(a) (i) The equation $(z+i)^7+(z-i)^7=0$ can be shown to have solutions satisfying $\frac{z+i}{z-i}=e^{\frac{2m+1}{7}\pi i}$ (do NOT prove this). Write down the set of values for m such that the equation has unique solutions where $\left(\frac{2m+1}{7}\right)\pi \in (-\pi,\pi]$. **1**

(ii) Given that $\frac{e^{2i\theta}+1}{e^{2i\theta}-1}=\frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}$, show that the solutions of $(z+i)^7+(z-i)^7=0$ can be written as $z=\cot\left(\frac{2m+1}{14}\pi\right)$ for these values of m. **3**

(iii) Using the binomial theorem, or otherwise, show that the non-zero roots of $(z+i)^7+(z-i)^7=0$ are the roots of $z^6-21z^4+35z^2-7=0$. **2**

(iv) Hence show that **2**

$$\sum_{k=0}^{2}\cot^2\left(\frac{2k+1}{14}\pi\right)=21$$

(b) A projectile of mass m is fired vertically upwards into a resistive medium, under the effect of gravity, with an initial velocity u. After reaching its maximum height H, it freefalls back to the ground, vertically. The object experiences a resistive force kv^2, when travelling in both directions, where v is the speed of the object and k is a positive constant.

The equation of motion for the object in freefall is given by $m\dot{v}=mg-kv^2$.

(i) Show that the terminal velocity v_τ that the object experiences when falling is **1**

$$v_\tau=\sqrt{\frac{mg}{k}}$$

(ii) Show that the time T at which the **maximum** height is reached is **3**

$$T=\frac{v_\tau}{g}\tan^{-1}\left(\frac{u}{v_\tau}\right)$$

(iii) Show that the **maximum** height H is given by **3**

$$H=\frac{v_\tau^2}{2g}\ln\left(1+\frac{u^2}{v_\tau^2}\right)$$

(iv) Hence show that the speed W on impact with the ground is **3**

$$W=\frac{u}{\sqrt{1+\frac{u^2}{v_\tau^2}}}$$

END OF EXAMINATION

Multiple Choice Answers

Question 1

This represents a circle in the complex plane with centre at $(1, -2)$ and radius 1.

The maximum distance from the origin to any point on the circle is the distance from the origin to the centre plus the radius.

1. Distance from origin to centre $(1, -2)$:

$$\sqrt{1^2 + (-2)^2} = \sqrt{5}$$

2. Maximum modulus of z:

$$\sqrt{5} + 1$$

Answer: (B)

Question 2

The magnitude of the vector is calculated as:

$$\sqrt{(\cos\theta)^2 + (\sin\theta)^2 + (\tan\theta)^2}$$

Simplifying:

$$\sqrt{\cos^2\theta + \sin^2\theta + \frac{\sin^2\theta}{\cos^2\theta}} = \sqrt{1 + \tan^2\theta}$$

Using the identity $1 + \tan^2\theta = \sec^2\theta$:

$$\sqrt{\sec^2\theta} = \sec\theta$$

Answer: (D)

Question 3

(C) "All ravens are black." must be false because the existence of even one non-black raven disproves this statement.

Answer: (C)

Question 4

Let's verify the collinearity of points A, B, and C for the different values of a and b given in the options. For each option, the point C is given as $\overrightarrow{OC} = 3\mathbf{i} + a\mathbf{j} + b\mathbf{k}$.

We want to check if vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, i.e., if one is a scalar multiple of the other.

1. Vector $\overrightarrow{AB}$:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j}) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

2. Vector $\overrightarrow{AC}$:

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3\mathbf{i} + a\mathbf{j} + b\mathbf{k}) - (\mathbf{i} + \mathbf{j}) = 2\mathbf{i} + (a-1)\mathbf{j} + b\mathbf{k}$$

Now, for collinearity, $\overrightarrow{AC} = \lambda\overrightarrow{AB}$ implies:

$$2\mathbf{i} + (a-1)\mathbf{j} + b\mathbf{k} = \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

Equating components, we get:

- $2 = \lambda \cdot 1$
- $a - 1 = \lambda \cdot (-2)$
- $b = \lambda \cdot 1$

Solving these equations:

- From $2 = \lambda$, we get $\lambda = 2$.
- For $a - 1 = -2\lambda$, substituting $\lambda = 2$, we have $a - 1 = -4$, so $a = -3$.
- For $b = \lambda$, substituting $\lambda = 2$, we have $b = 2$.

Thus, the correct values of a and b for the vectors to be collinear are $a = -3$ and $b = 2$.

Answer: (C)

Question 5

1. Calculate $\bar{z}$:

$$\bar{z} = \sqrt{3} - i$$

2. Calculate $\frac{\bar{z}}{z}$:

$$\frac{\bar{z}}{z} = \frac{\sqrt{3} - i}{\sqrt{3} + i} = \frac{(\sqrt{3} - i)^2}{(\sqrt{3})^2 + 1} = \frac{3 - 2\sqrt{3}i - 1}{4} = \frac{2 - 2\sqrt{3}i}{4} = \frac{1 - \sqrt{3}i}{2}$$

3. Effect on w:
Since $\frac{\bar{z}}{z}$ is a complex rotation, calculate the argument:

$$\text{Argument} = -\frac{\pi}{3}$$

Answer: (C)

Question 6

- (A) $\sim B \Rightarrow \sim A$ is the contrapositive and is always true.
- (B) A and $\sim B$ is false since $A \Rightarrow B$ implies B is true if A is true.

Answer: (B)

Question 7

1. Substitute $z = 1 + i$ in the polynomial:

$$(1+i)^3 + b(1+i)^2 + 6(1+i) - 4 = 0$$

2. Calculate powers:
   - $(1+i)^2 = 1 + 2i - 1 = 2i$ - $(1+i)^3 = (1+i)(2i) = 2i - 2 = -2 + 2i$

3. Substitute back:

$$(-2 + 2i) + b(2i) + 6 + 6i - 4 = 0$$

4. Separate real and imaginary parts:
   - Real: $-2 + 6 - 4 = 0$
   - Imaginary: $2 + 2bi + 6i = 0 \to b = -4$

Answer: (A)

Question 8

For object A:

$$h - \frac{1}{2}gt^2 = \text{height of } B$$

For object B:

$$v_0 t - \frac{1}{2}gt^2 = \text{height of } A$$

Setting both heights equal:

$$h = v_0 t \Rightarrow t = \frac{h}{v_0}$$

Answer: (B)

Question 9

The normal force N and frictional force F_f are dependent on the weight component perpendicular and parallel to the plane:

- $N = mg \cos\theta \leq mg$
- $F_f \leq \mu_s N \leq mg \sin\theta$

Answer: (A)

Question 10

To minimize the integral, differentiate and find critical points.

$$f(x) = \int_{-3}^{x^2-3x} e^{t^2}\, dt$$

Using the Fundamental Theorem of Calculus:

$$f'(x) = e^{(x^2-3x)^2} \cdot (2x - 3)$$

Setting $f'(x) = 0$:

$$2x - 3 = 0 \Rightarrow x = \frac{3}{2}$$

Answer: (B)

Extension 2, Task 4 (Trial), 2024: Question 11		
Suggested Solutions	**Marks**	**Marker's Comments**
Q11 (a) If $a = 3-5i$ and $b = 2+3i$ then $\bar{b} = 2-3i$ and $a-\bar{b} = 1-2i$	1	Aw1 Correct Answer
(b) If $z^2+2\bar{z}+3=0$, let $z = a+bi$ ie, $(a+bi)^2 + 2(a-bi)+3=0$ rearranging we have $a^2+2a-b^2+2b(a-1)i \equiv -3+0i$ equating real and imaginary parts $\left.\begin{array}{l} a^2+2a-b^2=-3 \\ 2b(a-1)=0 \end{array}\right\}$ $\therefore b=0, a=1$ when $b=0$ $a^2+2a+3=0$ $\Delta=-8<0$ and since $a \in \mathbb{R}$, reject $b=0$ when $a=1$ $b^2=6$ $\therefore b=\pm\sqrt{6}$ and $z = 1\pm\sqrt{6}i$ Aw1 Equating real and imaginary parts correctly. Aw2 Solving for a and b including reasoning as to why $b=0$ is rejected Aw3 Correct Solution	3	
(c) let $u=\tan x$ $\therefore \frac{du}{dx}=\sec^2 x$ $x=0, u=0$; $x=\frac{\pi}{4}, u=1$ $\tan^2 x+1=\sec^2 x$ $\therefore u^2+1=\sec^2 x$ so $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x\,dx$ $= \int_0^1 u^4(u^2+1)\,du$ $= \frac{12}{35}$	3	Aw1 correct $\frac{du}{dx}$ and new limits. Aw2. Correct integral in terms of u. Aw3. Correct Answer

Extension 2, Task 4 (Trial), 2024: Question 11		
Suggested Solutions	**Marks**	**Marker's Comments**
(d) $\int x^3\sqrt{1+x^2}\,dx$ $\quad u = 1+x^2 \Rightarrow x^2 = u-1$, $\quad du = 2x\,dx$ $= \frac{1}{2}\int (u-1)u^{\frac{1}{2}}\,du$ $= \frac{1}{2}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$ $= \frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}} + C$ $= \frac{1}{5}(1+x^2)^{\frac{5}{2}} - \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$	2	Aw1. Correct integral in terms of u. Aw2. Correct answer in terms of x.
(e)(i) $I_n = \int_0^{\pi} x^n \sin x\,dx$ $\quad u = x^n \quad v' = \sin x$, $\quad u' = nx^{n-1} \quad v = -\cos x$ $= -[x^n\cos x]_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x\,dx$ $= -[\pi^n\cos\pi - 0] + n[x^{n-1}\sin x]_0^{\pi} - n(n-1)\int_0^{\pi} x^{n-2}\sin x\,dx$ $\quad \cos\pi = -1$, $\quad u = x^{n-1} \quad v' = \cos x$, $\quad u' = (n-1)x^{n-2} \quad v = \sin x$ $= \pi^n + n[\pi^{n-1}\sin\pi - 0] - n(n-1)I_{n-2}$, $\quad \sin\pi = 0$ $\therefore I_n = \pi^n - n(n-1)I_{n-2}$ Aw1 Correct first application of integration by parts. Aw2. Evaluate $uv']_0^{\pi}$ <u>and</u> correct second application of integration by parts. Aw3 Arrive at I_n correctly.	3	

Extension 2, Task 4 (Trial), 2024: Question 11

Suggested Solutions	Marks	Marker's Comments
(e)(ii) $I_n = \pi^n - n(n-1)I_{n-2}$ $\therefore I_6 = \pi^6 - 6\cdot5\cdot I_4$ $I_4 = \pi^4 - 4\cdot3\cdot I_2$ $I_2 = \pi^2 - 2\cdot1\cdot I_0$ $I_0 = \int_0^\pi \sin x\,dx = -\cos x\Big]_0^\pi = 1-(-1) = 2$ $\therefore I_6 = \pi^6 - 30(\pi^4 - 12(\pi^2 - 2\times2))$ $= \pi^6 - 30\pi^4 + 360\pi^2 - 1440$	2	Aw1 $I_0 = 2$ Aw2 correct answer
(f)(i) $\dot{x}^2 = 4x - x^2$ $\frac{1}{2}\dot{x}^2 = 2x - \frac{1}{2}x^2$ $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right)$ $= 2 - x$ $= -1^2(x-2)$ which is in form $\ddot{x} = -n^2(x - x_0)$ which is simple harmonic motion	1	Aw1 arrive to form $\ddot{x} = -n^2(x - x_0)$
(f)(ii) By inspection, centre of motion is $x_0 = 2$ and $n = 1$, so $T = \frac{2\pi}{1} = 2\pi$ now at endpoints $\dot{x} = 0$ $\therefore 4x - x^2 = 0$ $x = 0, 4$ $\therefore$ Amplitude $= \frac{4-0}{2} = 2$	2	Aw1 correct x_0 and T Aw2 correct x_0, T and amplitude
(f)(iii) S.H.M $\therefore x = a\sin(nt + \alpha) + c$ when $t = 0$, $x = 4$ $\therefore 4 = 2\sin(\alpha) + 2$ and $x_0 = 2, a = 2, n = 1$ $\therefore \alpha = \frac{\pi}{2}$ $\therefore x = 2\sin(t + \frac{\pi}{2}) + 2$ $= 2\cos t + 2$	1	Aw1. Correct Answer

Extension 2, Trial HSC, 2024: Question 12

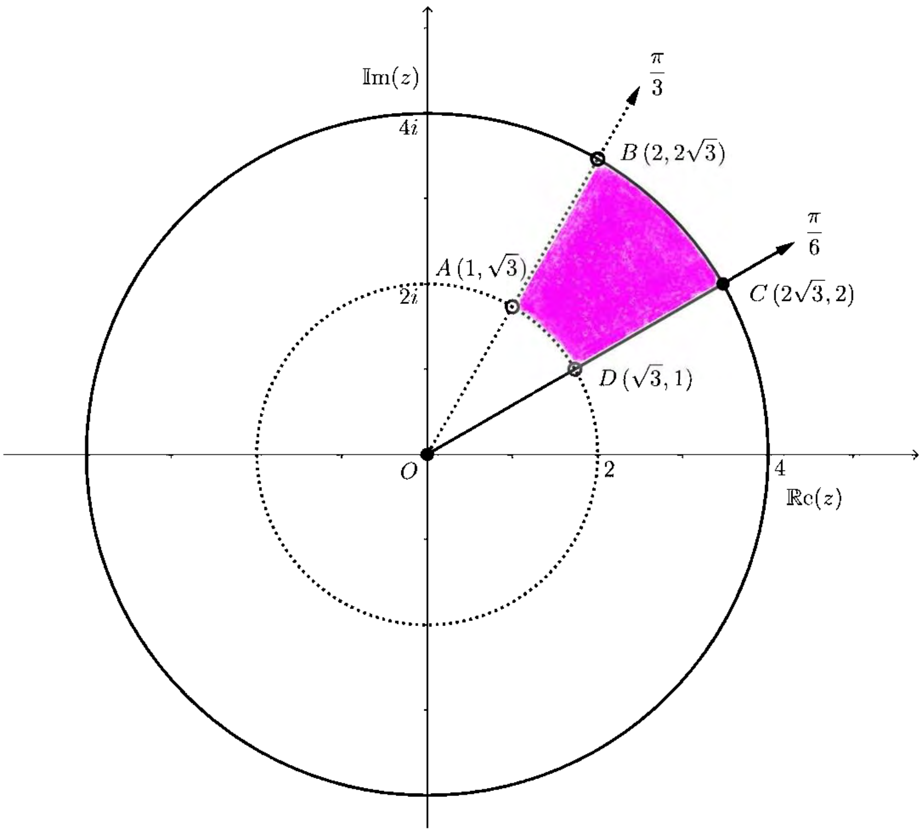
Suggested Solutions	Marks	Marker's Comments

(a) We are looking for the locus:

$$\{z \in \mathbb{C} | 2 < |z| \leq 4\} \cap \left\{z \in \mathbb{C} \middle| \frac{\pi}{6} \leq \arg(z) < \frac{\pi}{3}\right\}$$

and are required to determine all vertices.

The locus is



The points of intersection are, with respect to the diagram:

$$z_A = 2\exp\left(i\frac{\pi}{3}\right) = 1 + i\sqrt{3} \quad \rightarrow \quad (1, \sqrt{3})$$

$$z_B = 2z_A = 2 + 2i\sqrt{3} \quad \rightarrow \quad (2, 2\sqrt{3})$$

$$z_C = 4\exp\left(i\frac{\pi}{6}\right) = 2\sqrt{3} + 2i \quad \rightarrow \quad (2\sqrt{3}, 2)$$

$$z_D = \frac{1}{2}z_C = \sqrt{3} + i \quad \rightarrow \quad (\sqrt{3}, 1)$$

Marks: 3

Marker's Comments:

One mark for correct moduli and arguments.

One mark for correct boundaries of region + shading.

One mark for all vertices.

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(b) We have $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ unit vectors such that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$.

Consider then:

$$(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{0} \cdot \underset{\sim}{0}$$

The distributive properties of the dot product have that every term in the sum $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ forms an inner product with every term in the second bracket. Hence

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Note: Many students made the problem more complicated by appealing to vector geometry in lieu of using the algebraic properties of the dot product. This meant increased rate of error and complication.

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$$\begin{aligned}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) &= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} \\ &\quad + \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} \\ &\quad + \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{c}\end{aligned}$$

$$= a^2 + b^2 + c^2 + 2(\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c})$$

$$= 1 + 1 + 1 + 2(\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c}) \quad \text{(since each vector is unital)}$$

$$= 3 + 2(\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c})$$

But $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{0} \cdot \underset{\sim}{0} = 0^2 = 0$. Hence

$$0 = 3 + 2(\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c})$$

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First mark for logical move, either using dot product directly or vector geometry.

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which means

$$\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c} = -\frac{3}{2}$$

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Second mark for correct conclusion.

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(c)

(i) The sphere S_1 has centre $C(2,2,2)$ and passes through the point $A(4,4,4)$. We want the Cartesian equation of S_1.

We can find this using the Cartesian form directly, or by using the vector form.

We will do this first with Cartesian form: The sphere S_1 is the locus of all points from the point C having distance AC; that is,

$$S_1: (x-2)^2 + (y-2)^2 + (z-2)^2 = AC^2 = 3(4-2)^2 = 12$$

So

$$S_1: (x-2)^2 + (y-2)^2 + (z-2)^2 = 12$$

Otherwise, we may use the vector representation: Form the position vector

$$\overrightarrow{OR} = \overrightarrow{OC} + \overrightarrow{CA}$$

So, $\overrightarrow{OR}$ here maps from the origin of the coordinate system to the point A on the sphere. Then we have

$$\overrightarrow{OR} - \overrightarrow{OC} = \overrightarrow{CA}$$

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Part (i) could be handled from either Cartesian or vector standpoints.

Cartesian: First mark for knowledge of equation for sphere. Second mark for correct equation (i.e. determining the correct centre and radius).

Vector: First mark for correct quotation of vector form of a sphere. Second mark for correct application and final equation.

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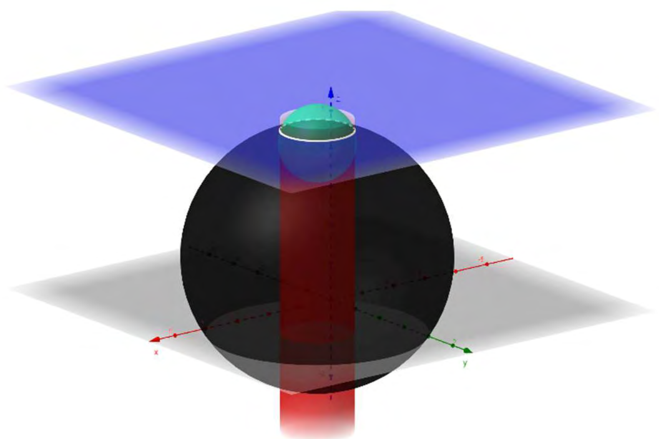
$$\Longrightarrow \begin{bmatrix} x-0 \\ y-0 \\ z-0 \end{bmatrix} - \begin{bmatrix} 2-0 \\ 2-0 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 4-2 \\ 4-2 \end{bmatrix}$$	1	
and we want all components x, y, z such that $$\left\lVert \begin{bmatrix} x-2 \\ y-2 \\ z-2 \end{bmatrix} \right\rVert = \left\lVert \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\rVert = \sqrt{2^2+2^2+2^2} = \sqrt{12}$$	1	
which yields an equivalent expression.		
(ii) Let $S_2 = (x-2)^2 + (y-2)^2 + (z-5)^2 = 1$. The circle of intersection is all $(x, y, z) \in S_1 \cap S_2$ at a given fixed z (else we have a cylinder).		First mark for correct move in solving simultaneously.
Since we have intersection iff x, y, z are common to both spheres, we have that $$1-(z-5)^2 = 12-(z-2)^2$$	1	Second mark for correct equation.
which gives $z = 16/3$. Then, in either S_1 or S_2 (choose S_2 WLOG), we have that $$(x-2)^2 + (y-2)^2 + \left(\frac{16}{3}-5\right)^2 = 1$$		It was not required that students state $z = 8/9$ alongside the equation as part of the solution since this (we will argue) can be implied from the calculation required to get to this point.
or equivalently, $$(x-2)^2 + (y-2)^2 = 8/9$$	1	But it should be noted: In 3-space, $(x-2)^2 + (y-2)^2 = \frac{8}{9}$ alone means the set of all triples $(x, y, z) \in \mathbb{R}^3$ such that $$(x-2)^2 + (y-2)^2 = \frac{8}{9}$$ for all $z \in \mathbb{R}$. This would produce a circular cylinder. That cylinder's boundary at $z = 16/3$ would coincide with the circle of intersection of S_1 and S_2.
The circle of intersection in $\mathbb{R}^3$ is therefore $$\mathcal{C}: \left\{ (x, y, 16/3) \in \mathbb{R}^3 \,\middle\vert\, (x-2)^2 + (y-2)^2 = \frac{8}{9} \right\}$$ 		

(d) (i) Consider $$(a-b)^2 \geq 0$$ $$\Leftrightarrow a^2 - 2ab + b^2 \geq 0$$ $$\Leftrightarrow \quad a^2 + b^2 \geq 2ab$$	1	Only one mark available.
(ii) $$a^2 + b^2 \geq 2ab \text{ (By result in (i))}$$ $$(a^2+b^2)(a+b) \geq 2ab(a+b)$$ $$a^3 + a^2b + b^2a + b^3 \geq 2a^2b + 2ab^2$$ $$a^3 + b^3 \geq a^2b + ab^2 \ldots (1)$$ Cyclic permutation of $a \mapsto b \mapsto c$ gives similarly: $$b^3 + c^3 \geq b^2c + bc^2 \quad \ldots (2)$$ $$c^3 + a^3 \geq c^2a + ca^2 \quad \ldots (3)$$ Add inequations (1) to (3): $$2(a^3+b^3+c^3) \geq a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2$$ $$\geq ab(a+b) + bc(b+c) + ca(c+a)$$ as required.	1 1	First mark for logical use of the result in part (i). Second mark for correct conclusion. If a candidate did not use the result in (i) at any stage, but proved the result, max. award was 1 (the question states 'Hence' as part of the requirement). If students used off-syllabus results without proof, max. award was 1 (e.g. using 3-case for AM-GM without proof).
(e) We have the initial conditions: $x = 0, \dot{x} = 2$. (i) From the equation of motion, $$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = x - 2 \Longrightarrow \frac{1}{2}\dot{x}^2 = \frac{1}{2}(x-2)^2 + \frac{C}{2}$$ Then $$\dot{x}^2 = (x-2)^2 + C$$	1	First mark for general solution to differential equation.
From the initial conditions, we evaluate: $4 = 4 + C \rightarrow C = 0$. So, the speed is given by the relation: $$\dot{x}^2 = (x-2)^2$$	1	Second mark for specialising solution via boundary condition.

(ii) The initial acceleration is $\ddot{x}(0) = 0 - 2 = -2\ \text{ms}^{-2}$ and initial velocity is $2\ \text{ms}^{-1}$. Hence, we move into the positive regime of the coordinate system with respect to displacement x. As $x \to 2^-$, we see • $\dot{x}^2 \to 0 \Longrightarrow \dot{x} \to 0$ • $\ddot{x} \to 0^-$ So, as we approach 2 from the left (i.e. $x \to 2^-$), both the acceleration and speed go to zero. If there is no acceleration, velocity can't change. Since the velocity is positive but approaching zero as acceleration approaches zero, the particle sees 2 as a limiting position. That is, as $t \to \infty$, $x \to 2^-$. Since $x \to 2^-$, we must have, given the particle moves only to the right (assuming right is positive) that $0 \leq x < 2$. Consequently, from the result in (i), we have *a priori* $$\dot{x} = \pm(x-2) \quad (\#)$$ Now, for sufficiently small time $t = \delta t$ after $t = 0$, we have the following conditions holding simultaneously: $$0 < x < 2 \quad (1)$$ $$\dot{x} > 0 \quad (2)$$ From (1), $x - 2 < 0$ and we need, by (2), that $\dot{x} > 0$, so we take the negative of (#): $$\dot{x} = -(x-2) = 2 - x$$	1 1	Part (ii) was not handled well. Students needed to first justify why the particle was bounded in the interval $[0,2)$ for all $t \geq 0$, then use this with initial conditions to justify the conclusion $\dot{x} = 2 - x$. This meant, students needed to <u>discuss</u> **both** the effects of $\dot{x}$ and $\ddot{x}$ (first mark). The second mark was obtained using the information now established to show that we must have only $\dot{x} = 2 - x$ as the solution. The depth of analysis here was not required; depth is provided for future understanding. Many students talked only of $\dot{x} \to 0$ or $\ddot{x} \to 0$ but not both. Showing $\dot{x} \to 0$ only without showing acceleration decays to zero means the system is left open to a possible non-zero acceleration if it reaches $\dot{x} = 0$, which means the particle will only momentarily stop before moving off again. Others saw velocity heading to zero and negative acceleration and claimed SHM as a consequence, or that the particle would start to move back to the origin after reaching 2. Those who stated SHM then went on to **correctly** derive the displacement equation…an equation containing **no** sinusoids. Always check your work to ensure

(iii) From (ii), $$\frac{dx}{dt} = 2 - x$$ so $$\int \frac{dx}{dt} \cdot \frac{1}{2-x} dt = \int dt$$ $$\Rightarrow \int \frac{dx}{2-x} = \int dt$$ $$-\log\lvert 2-x\rvert = t + C$$ Using the boundary conditions: $x = 0$ @ $t = 0$: $$C = -\log 2$$ So $$-\log\lvert 2-x\rvert = t - \log 2$$ Also, $0 \leq x < 2$, so $2 - x > 0$, hence $\log\lvert x-2\rvert = \log(x-2)$. We have then $$t = \log\left(\frac{2}{2-x}\right)$$ giving $$e^{-t} = \frac{2-x}{2}$$ so $$x = 2(1 - e^{-t})$$	1 1	consistency. If those who made the SHM mistake had done this, they could possibly have corrected their answer in this section. Part (iii) was handled very well in general. First mark for correct integration/general solution. Second mark for using boundary conditions to specialise the equation and make x the subject. Candidates did not have to justify that $\lvert 2-x\rvert = 2 - x$ here since it was implied by part (ii). That said, you should always state why the absolute value in the argument of a logarithm may be ignored. Must have given the conclusion in the form $x = x(t)$ (i.e. not leave in the form $t = t(x)$ after integration) since the question required so.

Student Number Solutions

Question 13

a) $|\vec{QS}|^2 = \vec{QS}\cdot\vec{QS}$

$= (\vec{AS}-\vec{AQ})\cdot(\vec{AS}-\vec{AQ})$ ✓ first dot product

$= |\vec{AS}|^2 - 2\vec{AQ}\cdot\vec{AS} + |\vec{AQ}|^2$

$= |\vec{AB}|^2 - 2|\vec{AB}||\vec{AC}|\cos\angle QAS + |\vec{AC}|^2$ $\left(\begin{array}{l}|\vec{AB}| = |\vec{AQ}| \\ |\vec{AC}| = |\vec{AS}|\end{array}\text{ given}\right)$

$= |\vec{AB}|^2 + 2|\vec{AB}||\vec{AC}|(-\cos(180^\circ - \angle BAC)) + |\vec{AC}|^2$

$= |\vec{AB}|^2 + 2|\vec{AB}||\vec{AC}|\cos\angle BAC + |\vec{AC}|^2$ ✓ handling angle

$= |\vec{AB}|^2 + 2\vec{AB}\cdot\vec{AC} + |\vec{AC}|^2$

$= (\vec{AB}+\vec{AC})\cdot(\vec{AB}+\vec{AC})$

$= (2\vec{AM})\cdot(2\vec{AM})$ ✓ finishing

i.e. $|\vec{QS}|^2 = 4|\vec{AM}|^2$

$|\vec{QS}| = 2|\vec{AM}|$ □

b) i) $|\underset{\sim}{u}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$ $\qquad |\underset{\sim}{v}| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$

$\therefore \hat{\underset{\sim}{u}} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$ ✓ $\qquad \hat{\underset{\sim}{v}} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$ ✓

ii) Let $\underset{\sim}{w} = \lambda\begin{pmatrix}1\\a\\b\end{pmatrix}$

$\underset{\sim}{w}\cdot\underset{\sim}{u} = 0 \Rightarrow 1+2a+3b=0$ —① ✓ enough simultaneous equations to give parallel vector

$\underset{\sim}{w}\cdot\underset{\sim}{v} = 0 \Rightarrow 1-2a+b=0$ —②

①+②: $4b = -2$

$b = -\frac{1}{2}$

sub into ①: $1+2a-\frac{3}{2}=0$

$a = \frac{1}{4}$

$\therefore \underset{\sim}{w} = \lambda\begin{pmatrix}1\\\frac{1}{4}\\-\frac{1}{2}\end{pmatrix}$

$|\hat{\underset{\sim}{w}}| = 1 \Rightarrow \lambda\sqrt{1^2+(\frac{1}{4})^2+(-\frac{1}{2})^2} = 1$

$\lambda\frac{\sqrt{21}}{4} = 1$

$\lambda = \frac{4}{\sqrt{21}}$

$\therefore \hat{\underset{\sim}{w}} = \frac{1}{\sqrt{21}}\begin{pmatrix}4\\1\\-2\end{pmatrix}$ ✓ answer

Student Number Solutions

Question 13 (cont.)

b) iii) $|\underset{\sim}{x}| = 1$

$|\underset{\sim}{x}|^2 = 1$

$(a\hat{\underset{\sim}{u}} + b\hat{\underset{\sim}{v}} + c\hat{\underset{\sim}{w}}) \cdot (a\hat{\underset{\sim}{u}} + b\hat{\underset{\sim}{v}} + c\hat{\underset{\sim}{w}}) = 1$ ✓ dot product

$a^2|\hat{\underset{\sim}{u}}|^2 + b^2|\hat{\underset{\sim}{v}}|^2 + c^2|\hat{\underset{\sim}{w}}|^2 + 2ab\,\hat{\underset{\sim}{u}}\cdot\hat{\underset{\sim}{v}} + 2ac\,\hat{\underset{\sim}{u}}\cdot\hat{\underset{\sim}{w}} + 2bc\,\hat{\underset{\sim}{v}}\cdot\hat{\underset{\sim}{w}} = 1$

$a^2.1 + b^2.1 + c^2.1 + 0 + 0 + 0 = 1$ ($\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{v}}, \hat{\underset{\sim}{w}}$ unit orthogonal vectors)

$a^2 + b^2 + c^2 = 1$ ☐ ✓ result

iv) $\cos\alpha = \dfrac{\hat{\underset{\sim}{u}} \cdot \underset{\sim}{x}}{1} = \hat{\underset{\sim}{u}} \cdot (a\hat{\underset{\sim}{u}} + b\hat{\underset{\sim}{v}} + c\hat{\underset{\sim}{w}})$

$= a|\underset{\sim}{u}|^2 + b\,\hat{\underset{\sim}{u}}\cdot\hat{\underset{\sim}{v}} + c\,\hat{\underset{\sim}{u}}\cdot\hat{\underset{\sim}{w}}$

$= a.1 + 0 + 0$ ($\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{v}}, \hat{\underset{\sim}{w}}$ unit orthoganal vectors)

i.e. $\cos\alpha = a$ ✓ showing one relationship

Similarly $\cos\beta = b$

$\cos\gamma = c$

and $\cos\alpha + \cos\beta + \cos\gamma = a + b + c$ ✓

c) i) ω is a root of $x^3 - 1 = 0$

$\therefore \omega^3 - 1 = 0$

$(\omega - 1)(\omega^2 + \omega + 1) = 0$

$\omega = 1$ or $\omega^2 + \omega + 1 = 0$ ✓

but $\omega \notin \mathbb{R}$ $\therefore$ ω is a root of $x^2 + x + 1 = 0$.

ii) Assume for sake of contradiction that

$(x+1)^{2n} + x^{2n} + 1 = (x^2 + x + 1)\,Q(x)$ where $n = 3k, k \in \mathbb{N}$

i.e. $(x+1)^{6k} + x^{6k} + 1 = (x^2 + x + 1)\,Q(x)$ ✓, $x \notin \mathbb{R}$

From i) if $x = \omega$: $(\omega+1)^{6k} + \omega^{6k} + 1 = \cancel{(\omega^2 + \omega + 1)}^{\,0}\,Q(\omega)$

$\omega^2 + \omega + 1 = 0 \Rightarrow (\omega + 1) = -\omega^2$

Student Number Solutions

Question 13 (cont.)

c) ii) $(-\omega^2)^{6k} + (\omega^3)^{2k} + 1 = 0$

$(\omega^4)^{3k} + (\omega^3)^{2k} + 1 = 0$

$(\omega^3)^{4k} + (\omega^3)^{2k} + 1 = 0$ ✓

$1 + 1 + 1 = 0$ as $\omega^3 = 1$ from i)

this is a contradiction and so

$(x+1)^{2n} + x^{2n} + 1$ is NOT divisible by x^2+x+1 if n is divisible by 3.

d) Consider S_n: $\frac{1}{2} + \cos\theta + \cos 2\theta + \dots + \cos(n-1)\theta = \dfrac{\sin\left(\frac{2n-1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$

for $n \geq 2$, $n \in \mathbb{Z}$, $\theta \neq 2k\pi$, $k \in \mathbb{Z}$

Consider S_2: $\frac{1}{2} + \cos\theta = \dfrac{\sin\frac{3\theta}{2}}{2\sin\frac{\theta}{2}}$

$$\text{RHS} = \frac{\sin\left(\theta + \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\theta\cos\frac{\theta}{2} + \cos\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$ ✓ expand sine of sum

$$= \frac{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + \frac{\cos\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{1}{2}\left(2\cos^2\frac{\theta}{2} - 1 + 1\right) + \frac{\cos\theta}{2}$$

$$= \frac{1}{2}(\cos\theta + 1) + \frac{1}{2}\cos\theta$$ ✓ finish

$$= \frac{1}{2} + \cos\theta$$

$$= \text{LHS}$$

$\therefore$ S_2 is true

Question 13 (cont.)

d) cont. Consider S_k: $\frac{1}{2}+\cos\theta+\cos 2\theta+\ldots+\cos(k-1)\theta = \dfrac{\sin\left(\frac{2k-1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$

Consider also S_{k+1}: $\frac{1}{2}+\cos\theta+\cos 2\theta+\ldots+\cos(k-1)\theta+\cos k\theta = \dfrac{\sin\left(\frac{2k+1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$

With S_{k+1}:

$$LHS = \frac{1}{2}+\cos\theta+\ldots+\cos(k-1)\theta+\cos k\theta$$

$$= \frac{\sin\left(\frac{2k-1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + \cos k\theta$$, if S_k is true

✓ substitute S_k result

$$= \frac{\sin\left(k\theta-\frac{\theta}{2}\right)+2\cos k\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin k\theta\cos\frac{\theta}{2} - \cos k\theta\sin\frac{\theta}{2} + 2\cos k\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin k\theta\cos\frac{\theta}{2} + \cos k\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(k\theta+\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

✓ finish

$$= \frac{\sin\left(\frac{2k+1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

$= RHS$

$\therefore$ S_{k+1} is true if S_k is true. Since S_2 is true, $S_{2+1}=S_3$ is true, $S_{3+1}=S_4$ is true and so on for all S_n by the principle of mathematical induction.

Question 14

a)

i.

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$
$$= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{AB} + i\overrightarrow{AB}\right)$$

1 mark for recognising $\overrightarrow{AX} = \frac{1}{2}\left(\overrightarrow{AB} + i\overrightarrow{AB}\right)$

$$= 2a + \frac{1}{2}(2b + 2ib)$$
$$= 2a + b + ib$$

1 mark for completing the proof.

Therefore X can be represented by $2a + b + ib$

ii.

$$\overrightarrow{OW} = \frac{1}{2}(2a + i2a)$$
$$= a + ai$$

$$\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BY}$$
$$= 2a + 2b + \frac{1}{2}(2c + 2ci)$$
$$= 2a + 2b + c + ic \text{ or } -2d - c + ic$$

$$\overrightarrow{OZ} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CZ}$$
$$= 2a + 2b + 2c + \frac{1}{2}(d + id)$$
$$= 2a + 2b + 2c + d + id \text{ or } -d + id$$

$$\overrightarrow{WY} = \overrightarrow{OY} - \overrightarrow{OW}$$
$$= (2a + 2b + c + ic) - (a + ai)$$
$$= (a + 2b + c) + (c - a)i \text{ or } (b - d) + (c - a)i$$

$$\overrightarrow{XZ} = \overrightarrow{OZ} - \overrightarrow{OX}$$
$$= (2a + 2b + 2c + d + id) - (2a + b + ib)$$
$$= (b + 2c + d) + (d - b)i \text{ or } (c - a) + (d - b)i$$

Method 1:

$$i\overrightarrow{XZ} = (b - d) + (b + 2c + d)i$$

$$\overrightarrow{WY} - i\overrightarrow{XZ} = (a + 2b + c) + (c - a)i - (b - d) - (b + 2c + d)i$$
$$= (a + b + c + d) - (a + b + c + d)i$$
$$= 0 \quad (2a + 2b + 2c + 2d = 0 \Rightarrow a + b + c + d = 0)$$

$$\therefore \overrightarrow{WY} = i\overrightarrow{XZ}$$

Therefore WY and XZ are equal in length and are perpendicular.

Method 2:

$$\left|\overrightarrow{WY}\right| = |(a+2b+c)+(c-a)i|$$

$$= |(b-d)+(c-a)i| \qquad (a+b+c+d=0)$$

$$= \sqrt{(b-d)^2+(c-a)^2}$$

$$= \sqrt{(c-a)^2+(b-d)^2}$$

$$= |(c-a)+(d-b)i|$$

$$= |(b+2c+d)+(d-b)i| \qquad (a+b+c+d=0)$$

$$= \left|\overrightarrow{XZ}\right|$$

$$\overrightarrow{WY}\cdot\overrightarrow{XZ} = \begin{pmatrix} b-d \\ c-a \end{pmatrix}\cdot\begin{pmatrix} c-a \\ d-b \end{pmatrix}$$

$$= (b-d)(c-a)+(c-a)(d-b)$$

$$= (b-d)(c-a)-(b-d)(c-a)$$

$$= 0$$

$\therefore \overrightarrow{WY} \perp \overrightarrow{XZ}$

Therefore the intervals WY and XZ are perpendicular and equal in magnitude.

1 mark for working out one of W, Y or Z

1 mark for working out the remaining two of W, Y or Z

1 mark completing the proof

b)

Test for $n = 2$:

$$a_{2-1} = a_1$$
$$= 0$$
$$< \frac{1}{2}$$
$$= a_2$$

$\therefore$ the statement is true for $n = 2$.

Assume the statement is true for $n = k, k \in \mathbb{Z}, k > 1$.

$\therefore a_{k-1} < a_k$

Test for $n = k + 1$

RTP: $a_k < a_{k+1}$

Method 1

$$a_{k+1} = \frac{1 + a_k}{2 + a_k}$$
$$= 1 - \frac{1}{2 + a_k}$$
$$> 1 - \frac{1}{2 + a_{k-1}}$$
$$= \frac{2 + a_{k-1} - 1}{2 + a_{k-1}}$$
$$= \frac{1 + a_{k-1}}{2 + a_{k-1}}$$
$$= a_k$$

$a_k > a_{k-1}$ (by assumption)

$$2 + a_k > 2 + a_{k-1}$$
$$\frac{1}{2 + a_k} < \frac{1}{2 + a_{k-1}}$$
$$-\frac{1}{2 + a_k} > -\frac{1}{2 + a_{k-1}}$$

Note: The steps above should be shown, as it can be seen that the inequality sign had to flip multiple times.

$\therefore$ if the statement is true for $n = k$, then it is true for $n = k + 1$

$\therefore$ the statement is true for $n > 1, n \in \mathbb{Z}$ by the principle of mathematical induction

1 mark for proving base case

1 mark for using the assumption appropriately in the proof

1 mark for progress towards the result after using the assumption

1 mark for completing the proof

Method 2

RTP: $a_{k+1} - a_k > 0$

$$\text{LHS} = \frac{1 + a_k}{2 + a_k} - \frac{1 + a_{k-1}}{2 + a_{k-1}}$$

$$= \frac{(1 + a_k)(2 + a_{k-1}) - (1 + a_{k-1})(2 + a_k)}{(2 + a_k)(2 + a_{k-1})}$$

$$= \frac{(2 + a_{k-1} + 2a_k + a_k a_{k-1}) - (2 + a_k + 2a_{k-1} + a_k a_{k-1})}{(2 + a_k)(2 + a_{k-1})}$$

$$= \frac{a_k - a_{k-1}}{(2 + a_k)(2 + a_{k-1})}$$

$$> \frac{a_k - a_k}{(2 + a_k)(2 + a_{k-1})} \qquad \text{(by assumption)}$$

$$= 0$$

$\therefore a_k < a_{k+1}$

$\therefore$ if the statement is true for $n = k$, then it is true for $n = k + 1$

$\therefore$ the statement is true for $n > 1, n \in \mathbb{Z}$ by the principle of mathematical induction

1 mark for proving base case

1 mark for using the assumption appropriately in the proof

1 mark for progress towards the result after using the assumption

1 mark for completing the proof

Any other method leads to a set of working out that gets you no where, hence no furhter marks awarded after the base case.

c)

i.

Let $\frac{1}{x(1+x^2)} \equiv \frac{a}{x} + \frac{bx+c}{1+x^2}$

Method 1:

$a(1+x^2) + (bx+c)x \equiv 1$

$x = i \Rightarrow -b + ci = 1$

Equating real component $\Rightarrow b = -1$

Equating imaginary component $\Rightarrow c = 0$

$\therefore a(1+x^2) - x^2 = 1$

$x = 1 \Rightarrow 2a - 1 = 1$

$a = 1$

$\therefore \frac{1}{x(1+x^2)} \equiv \frac{1}{x} - \frac{x}{1+x^2}$

Method 2:

$a(1+x^2) + (bx+c)x \equiv 1$

$a + ax^2 + bx^2 + cx \equiv 1$

$(a+b)x^2 + cx + a \equiv 1$

Equating the constant term $\Rightarrow a = 1$

Equating the coefficient of $x \Rightarrow c = 0$

Equating the coeffiient of $x^2 \Rightarrow a + b = 0 \Rightarrow b = -1$ (since $a = 1$)

1 mark for working out one of the constants

1 mark for completing the rest of the proof.

ii.

$$\int \frac{\tan^{-1} x}{x^2} dx = \int \frac{d}{dx}\left(-\frac{1}{x}\right) \tan^{-1} x \; dx$$

$$= \left(-\frac{\tan^{-1} x}{x}\right) + \int \frac{1}{x(1+x^2)} dx$$

$$= \left(-\frac{\tan^{-1} x}{x}\right) + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$$

$$= -\frac{\tan^{-1} x}{x} + \ln|x| - \frac{1}{2}\ln|1+x^2| + C$$

1 mark for applying integration by parts correctly

1 mark for completing the rest of the answer.

d)

i.

$$\sum_{k=0}^{n}(-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$$

$$= \frac{1(1-(-x)^{n+1})}{1-(-x)} \quad \text{(sum of a geometric series with } a = 1, r = -x, \text{ of } n+1 \text{ terms)}$$

$$= \frac{(1-(-x)^{n+1})}{1+x}$$

$$= \frac{1}{1+x} - \frac{(-x)^{n+1}}{1+x}$$

$$= \frac{1}{1+x} - \frac{(-1)^{n+1}x^{n+1}}{1+x}$$

1 mark

ii.

$\frac{x^{n+1}}{1+x} \geq 0$ for $x \in [0,1]$ (The domain in which this is true MUST be a acknowledged!!)

$$\therefore \int_0^1 \frac{x^{n+1}}{1+x}\,dx \geq 0$$

LHS inequality proven. 1 mark

$x \in [0,1] \Rightarrow 1 + x \in [1,2]$

$\therefore \frac{x^{n+1}}{1+x} \leq x^{n+1}$ for $x \in [0,1]$ (The domain in which this is true MUST be a acknowledged!!)

$$\int_0^1 \frac{x^{n+1}}{1+x}\,dx \leq \int_0^1 x^{n+1}\,dx$$

$$= \left[\frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \frac{1}{n+2}$$

RHS inequality proven

$$\therefore 0 \leq \int_0^1 \frac{x^{n+1}}{1+x}\,dx \leq \frac{1}{n+2}$$

1 mark

iii.

$$\int_0^1 \sum_{k=0}^{n} (-1)^k x^k \, dx = \int_0^1 \frac{1}{1+x} dx - \int_0^1 \frac{(-1)^{n+1} x^{n+1}}{1+x} dx$$

$$\left[\sum_{k=0}^{n} \frac{(-1)^k x^{k+1}}{k+1}\right]_0^1 = [\ln|1+x|]_0^1 - (-1)^{n+1} \int_0^1 \frac{x^{n+1}}{1+x} dx$$ 1 mark for integrating both sides

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} = \ln 2 - (-1)^{n+1} \int_0^1 \frac{x^{n+1}}{1+x} dx$$

Note: Many students integrated without limits and then subsituted $x = 1$ in,

doing that is not mathematically correct because you neglected the constant!

There was no penalty, but something to take note of.

Now,

$$0 \le \int_0^1 \frac{x^{n+1}}{1+x} dx \le \frac{1}{n+2} \quad (\text{Proven in part ii.})$$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \int_0^1 \frac{x^{n+1}}{1+x} dx \le \lim_{n\to\infty} \frac{1}{n+2}$$

$$\therefore \lim_{n\to\infty} \int_0^1 \frac{x^{n+1}}{1+x} dx = 0 \quad (\text{Squeeze Theorem})$$

Note: It is **vital** that student acknowledge we are trying to prove an equality statement,

simply stating that because $\lim_{n\to\infty} \frac{1}{n+2} = 0$, then $\int_0^1 \frac{x^{n+1}}{1+x} dx = 0$ is insufficient.

You must acknowledge the sandwich theorem, or the equivalent idea, i.e. $0 \le \int_0^1 \frac{x^{n+1}}{1+x} dx \le 0$,

before making the conclusion that $\int_0^1 \frac{x^{n+1}}{1+x} dx = 0$.

$$\therefore \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} = \ln 2 - \lim_{n\to\infty} (-1)^{n+1} \int_0^1 \frac{x^{n+1}}{1+x} dx$$

$$= \ln 2 - 0$$

$$= \ln 2$$ 1 mark for completing the proof

Start here for Question Number: 15

(a)

(i) $(z+i)^7 + (z-i)^7 = 0$

$\left(\frac{z+i}{z-i}\right)^7 = -1 = e^{(2m+1)\pi i}, \quad m \in \mathbb{Z}$

$\frac{z+i}{z-i} = e^{\left(\frac{2m+1}{7}\right)\pi i}$

$-\pi < \frac{2m+1}{7}\pi \leq \pi$

$-7 < 2m+1 \leq 7$

$-8 < 2m \leq 6$

$-4 < m \leq 3$ m is an integer

$m = -3, -2, -1, 0, 1, 2, 3$ 1 mark

(ii) $\frac{z+i}{z-i} = e^{i\theta}$ where $\theta = \left(\frac{2m+1}{7}\right)\pi$

$z + i = e^{i\theta}(z-i)$

$z(1-e^{i\theta}) = -i(1+e^{i\theta})$

$z = i\frac{e^{i\theta}+1}{e^{i\theta}-1} = i\frac{e^{i\theta/2}+e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} = \frac{\frac{e^{i\theta/2}+e^{-i\theta/2}}{2}}{\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}} = \frac{\cos\theta/2}{\sin\theta/2}$ ✓ ✓

$= \cot(\theta/2)$ ✓

$\therefore z = \cot\left(\frac{2m+1}{14}\pi\right), \quad m = \pm 3, \pm 2, \pm 1, 0$

1 mark for obtaining expression for z

1 mark for using the half angle formula

1 mark for using the expression for ~~cos θ and sin θ~~ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

(iii) $(z+i)^7 + (z-i)^7 = 0$

$$\sum_{k=0}^{7} {}^7C_k z^k i^{7-k} + \sum_{k=0}^{7} {}^7C_k z^k (-i)^{7-k} = 0$$

$$\sum_{k=0}^{7} {}^7C_k z^k i^{7-k} \left(1 + (-1)^{7-k}\right) = 0$$

$k=0$, $(-1)^{7-k} = -1$
$k=1$ 1
$k=2$ -1
$k=3$ 1
$k=4$ -1
$k=5$ 1
$k=6$ -1
$k=7$ 1

${}^7C_1 = 7$

${}^7C_3 = \frac{7!}{3!4!} = \frac{7\cdot6\cdot5}{6} = 35$

${}^7C_5 = \frac{7!}{5!2!} = 21$

1 mark for ~~expanding~~
1 mark ~~and~~ for simplifying ✓

$$\sum_{k=0}^{7} {}^7C_k z^k i^{7-k} \left(1 + (-1)^{7-k}\right) = 0$$

$${}^7C_1 z^1 i^6 \times 2 + {}^7C_3 z^3 i^4 (2) + {}^7C_5 z^5 i^2 (2) + {}^7C_7 z^7 \times 2 = 0$$

$$-14z + 70z^3 - 42z^5 + 2z^7 = 0$$

$$2z(z^6 - 21z^4 + 35z^2 - 7) = 0$$

$z = 0$ or $z^6 - 21z^4 + 35z^2 - 7 = 0$ ✓

Alternative approach

$z^7 + 7iz^6 - 21z^5 - 35iz^4 + 35z^3 + 21iz^2 - 7z - i +$
$z^7 - 7iz^6 - 21z^5 + 35iz^4 + 35z^3 - 21iz^2 - 7z + i = 0$
$2z^7 - 42z^5 + 70z^3 - 14z = 0$
$2z(z^6 - 21z^4 + 35z^2 - 14) = 0$

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Additional writing space on back page.

(iii) $z^6 - 21z^4 + 35z^2 - 7 = 0$

$(z^2)^3 - 21(z^2)^2 + 35z^2 - 7 = 0$ ← if you are using this approach the first mark is to mention that for k = 3, $\cot^2\frac{\pi}{2} = 0$ is not a solution. The second mark is to $\sum_{k=0}^{2} \cot^2\left(\frac{2k+1}{14}\pi\right) = 21$

$$\sum_{k=0}^{2} \cot^2\left(\frac{2k+1}{14}\pi\right) = 21$$

$k = 0$, $\cot^2\frac{\pi}{14}$

$k = 1$ $\cot^2\frac{3\pi}{14}$ $k = -1$, $\cot^2\left(\frac{-\pi}{14}\right) = \cot^2(\pi/14)$

$k = 2$ $\cot^2\left(\frac{5\pi}{14}\right)$ $k = -2$, $\cot^2\left(\frac{-3\pi}{14}\right) = \cot^2\left(\frac{3\pi}{14}\right)$

$k = 3$ $\cot^2\left(\frac{\pi}{2}\right) = 0$ is not a solution. $k = -3$, $\cot^2\left(-\frac{5\pi}{14}\right) = \cot^2\left(\frac{5\pi}{14}\right)$

Non Zero Solutions: $\frac{\pi}{14}, \frac{-\pi}{14}, \frac{3\pi}{14}, \frac{-3\pi}{14}, \frac{5\pi}{14}, \frac{-5\pi}{14}$ for $z^6 - 21z^4 + 35z^2 - 7 = 0$

$-\cot^2(\pi/14) + \cot\frac{\pi}{14}\left(\cot\frac{3\pi}{14} - \cot\frac{3\pi}{14} + \cot\frac{5\pi}{14} - \cot\left(\frac{5\pi}{14}\right)\right)$

$-\cot\frac{\pi}{14}\left(\cot\frac{3\pi}{14} - \cot\frac{3\pi}{14} + \cot\frac{5\pi}{14} - \cot\left(\frac{5\pi}{14}\right)\right)$ ✓ for equating

$-\cot^2\frac{3\pi}{14} + \cot\frac{3\pi}{14}\left(\cot\frac{5\pi}{14} - \cot\frac{5\pi}{14}\right)$

$-\cot\frac{3\pi}{14}\left(\cot\frac{5\pi}{14} - \cot\frac{5\pi}{14}\right) - \cot^2\left(\frac{5\pi}{14}\right) = -21$

$\cot^2\frac{\pi}{14} + \cot^2\frac{3\pi}{14} + \cot^2\frac{5\pi}{14} = 21.$ ✓ for simplifying

$(\Sigma t)^2 = \Sigma t^2 + 2\Sigma t_i t_j$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

b	(i)	The equation of motion for the falling object is $m\dot{v} = mg - kv^2$. Terminal velocity is reached when there is no more acceleration. Then $v_\tau^2 = \frac{mg}{k} \to v_\tau = \sqrt{\frac{mg}{k}}$. ✓
	(ii)	When the object rises, the equation of motion is

$$m\dot{v} = -mg - kv^2$$

Then

$$m\frac{dv}{dt} = -(mg + kv^2)$$

$$\frac{m}{mg + kv^2}dv = -dt$$

so

$$\int \frac{m}{mg + kv^2}dv = -\int dt \quad \text{...(1)}$$

Now

$$= \sqrt{\frac{mg}{k}} \cdot \frac{1}{g} \int \frac{d\left(\frac{v}{v_\tau}\right)}{1 + \left(\frac{v}{v_\tau}\right)^2}$$

$$= \frac{v_\tau}{g}\tan^{-1}\left(\frac{v}{v_\tau}\right) + C$$ ✓

Hence, by (1),

$$\frac{v_\tau}{g}\tan^{-1}\left(\frac{v}{v_\tau}\right) + C = -t$$

At $t = 0, v = u$, so

$$C = -\frac{v_\tau}{g}\tan^{-1}\left(\frac{u}{v_\tau}\right)$$ ✓

Hence

$$t = \frac{v_\tau}{g}\left(\tan^{-1}\left(\frac{u}{v_\tau}\right) - \tan^{-1}\left(\frac{v}{v_\tau}\right)\right)$$

When maximum height is reached, $v = 0$, so

$$T = \frac{v_\tau}{g}\left(\tan^{-1}\left(\frac{u}{v_\tau}\right) - \tan^{-1}\left(\frac{0}{v_\tau}\right)\right) = \frac{v_\tau}{g}\tan^{-1}\left(\frac{u}{v_\tau}\right)$$ ✓

For (ii) if you use the wrong equation:

ie: $m\dot{v} = mg - kv^2$

you would not be able to obtain the required result.

If you obtain the final result, you would not receive any marks in general.

(iii) For maximum height, equation of motion again is

$$m\dot{v} = -mg - kv^2$$

Then

$$mv\frac{dv}{dx} = -(mg + kv^2)$$

so

$$\int \frac{mv}{mg + kv^2}dv = -\int dx$$

Now

$$\int \frac{mv}{mg + kv^2}dv = \frac{m}{2k}\int \frac{2kv}{mg + kv^2}dv$$

$$= \frac{m}{2k}\ln|mg + kv^2| + C$$

so

$$\frac{m}{2k}\ln|mg + kv^2| + C = -x$$ ✓

At $x = 0, v = u$, so

$$C = -\frac{m}{2k}\ln(mg + ku^2)$$

Hence

$$x = \frac{m}{2k}\ln\left[\frac{mg + ku^2}{mg + kv^2}\right]$$ ✓

Maximum height is reached when $v = 0$, so

$$H = \frac{m}{2k}\ln\left(\frac{mg + ku^2}{mg}\right)$$

$$= \frac{mg}{(2g)k}\ln\left(1 + \left(\frac{k}{mg}\right)u^2\right)$$ ✓

$$= \frac{v_\tau^2}{2g}\ln\left(1 + \frac{u^2}{v_\tau^2}\right)$$

For (iii) Same remark regarding using the wrong equation.

(iv) Now find the velocity on impact. Re-initialise measurement as starting from $x = 0$, downward motion is taken to be positive. Then the equation of motion when in freefall is:

$$m\dot{v} = mg - kv^2$$

so

$$mv\frac{dv}{dx} = mg - kv^2$$

so

$$\frac{mv}{mg - kv^2}dv = dx$$

so

$$-\frac{m}{2k}\cdot\frac{-2kv}{mg - kv^2}dv = dx$$

Hence

$$-\frac{m}{2k}\int\frac{-2kv}{mg - kv^2}dv = \int dx$$

$$-\frac{m}{2k}\ln|mg - kv^2| = x + C$$

When $x = 0, v = 0$, so

$$-\frac{m}{2k}\ln(mg) = C$$

So

$$x = \frac{m}{2k}(\ln(mg) - \ln|mg - kv^2|)$$

$$= \frac{m}{2k}\ln\left|\frac{mg}{mg - kv^2}\right|$$

Now, when $v = 0$, there is a net acceleration downward of $+g$, so object will move in that direction. As velocity increases, $mg - kv^2 \to 0$, but always $mg > kv^2$, so $\left|\frac{mg}{mg-kv^2}\right| = \left(\frac{mg}{mg-kv^2}\right)$. Hence

$$\frac{2k}{m}x = \ln\left(\frac{mg}{mg - kv^2}\right)$$

Velocity W at $x = H$ is

$$\frac{2k}{m}\cdot\frac{v_\tau^2}{2g}\ln\left(1 + \frac{u^2}{v_\tau^2}\right) = \ln\left(\frac{mg}{mg - kW^2}\right)$$

$$\therefore \ln\left(1 + \frac{u^2}{v_\tau^2}\right) = \ln\left(\frac{mg}{mg - kW^2}\right)$$

so, dividing LHS argument by mg and using $\frac{mg}{k} = v_\tau^2$:

$$1 + \frac{u^2}{v_\tau^2} = \frac{1}{1 - \frac{W^2}{v_\tau^2}}$$

$$1 - \frac{W^2}{v_\tau^2} + \frac{u^2}{v_\tau^2} - \frac{W^2}{v_\tau^2}\cdot\frac{u^2}{v_\tau^2} = 1$$

so

$$W^2\left(1 + \frac{u^2}{v_\tau^2}\right) = u^2$$

so

$$W^2 = \frac{u^2v_\tau^2}{v_\tau^2 + u^2} = \frac{u^2}{1 + u^2/v_\tau^2}$$

so

$$W = \frac{u}{\sqrt{1 + u^2/v_\tau^2}}$$

with $W > 0$ since velocity is in the positive direction.